(36) QN.→ verify Euler's theorem when

$$u = x^3 \log \frac{y}{x}.$$

Ans.→ Here u is homogeneous function

of x and y of degree 3

Hence to verify Eulers theorem

I have to prove, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

$$u = x^3 \log\frac{y}{x} \quad — (A)$$

Diff. partially w.r.t. x keeping y constant

$$\frac{\partial u}{\partial x} = 3x^2(\log y - \log x) - x^3 \cdot \frac{1}{x}$$

$$x\frac{\partial u}{\partial x} = 3x^3(\log y - \log x) - x^3 \quad — (1)$$

Again diff (A) partially w.r.t. y keeping x constant

$$\frac{\partial u}{\partial y} = x^3 \times \frac{1}{\frac{y}{x}} \times \frac{1}{x} \times 1$$

$$\frac{\partial u}{\partial y} = x^3 \times \frac{1}{y}$$

$$y\frac{\partial u}{\partial y} = x^3 \quad — (2)$$

(1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3(\log y - \log x) - \cancel{x^3} + \cancel{x^3} = 3u$$

$$u = x^3 \log\frac{y}{x} = x^3 \phi\left(\frac{y}{x}\right)$$

Here u is a homogeneous function of degree 3.

Hence the given function satisfies Eulers theorem.

(4) If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ prove that

$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z)$.

Ans. → $\because \frac{x^2}{a^2+u}$

$$\therefore \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \quad \text{--- (1)}$$

Diff. (1) partially w.r.t. x, keeping y and z con

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$$

Diff. P. w.r.t. x, keeping y and z constant.

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} + y^2 \times \frac{-1}{(b^2+u)^2}\frac{\partial u}{\partial x} + z^2 \times \frac{-1}{(c^2+u)^2}\frac{\partial u}{\partial x}$$

$$\text{or, } \frac{2x}{a^2+u} = \frac{\partial u}{\partial x}\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right) \quad \text{--- (1)}$$

Similarly

$$\frac{2y}{b^2+u} = \frac{\partial u}{\partial y}\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right) \quad \text{--- (2)}$$

$$\frac{2z}{c^2+u} = \frac{\partial u}{\partial z}\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right) \quad \text{--- (3)}$$

multiplying (1) by x, (2) by y and (3) by z and then adding.

$$2\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right) = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

Again squaring and adding ①, ② and ③, we have,

$$\frac{4x^2}{(a^2+u)^2}+\frac{4y^2}{(b^2+u)^2}+\frac{4z^2}{(c^2+u)^2}=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2$$

$$\text{or, } 4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{4z^2}{(c^2+u)^2}\right]=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2$$

$$4=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$4=(ux^2+uy^2+uz^2)\times\frac{2}{xux+yuy+zuz}$$

or, $2(x\,ux+y\cdot uy+z\cdot uz)=ux^2+uy^2+uz^2$ proved.

(40) If $u=F(x-y,\ y-z,\ z-x)$ and $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ all exist, prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

Ans.→ $\because u=F(x-y,\ y-z,\ z-x)$

Let, $X=x-y$

$Y=y-z$

$Z=z-x$

$\therefore \frac{\partial X}{\partial x}=1$	$\frac{\partial X}{\partial y}=-1$	$\frac{\partial X}{\partial z}=0$
$\frac{\partial Y}{\partial x}=0$	$\frac{\partial Y}{\partial y}=1$	$\frac{\partial Y}{\partial z}=-1$
$\frac{\partial Z}{\partial x}=-1$	$\frac{\partial Z}{\partial y}=0$	$\frac{\partial Z}{\partial z}=1$

$\because u = F(X, Y, Z)$

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial F}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial F}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= \frac{\partial F}{\partial X}\cdot 1 + \frac{\partial F}{\partial Y}\cdot 0 + \frac{\partial F}{\partial Z}\cdot(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial X} - \frac{\partial F}{\partial Z} \quad \text{— (1)}$$

Again, $$\frac{\partial u}{\partial y} = \frac{\partial F}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial F}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial F}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$

$$= \frac{\partial F}{\partial X}\cdot(-1) + \frac{\partial F}{\partial Y}\cdot 1 + \frac{\partial F}{\partial Z}\cdot 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial F}{\partial X} + \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Y} - \frac{\partial F}{\partial X} \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial F}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial F}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial F}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$$= \frac{\partial F}{\partial X}\cdot 0 + \frac{\partial F}{\partial Y}\cdot(-1) + \frac{\partial F}{\partial Z}\times 1$$

$$\frac{\partial u}{\partial z} = 0 - \frac{\partial F}{\partial Y} + \frac{\partial F}{\partial Z} = \frac{\partial F}{\partial Z} - \frac{\partial F}{\partial Y} \quad \text{— (3)}$$

(1) + (2) + (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial F}{\partial X}} - \cancel{\frac{\partial F}{\partial Z}} + \cancel{\frac{\partial F}{\partial Y}} - \cancel{\frac{\partial F}{\partial X}} + \cancel{\frac{\partial F}{\partial Z}} - \cancel{\frac{\partial F}{\partial Y}}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ proved.